Proceedings of the ASME 2014 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2014
August 17-20, 2014, Buffalo, New York, USA

DETC2014-34713

A UNIFIED CONCEPT FOR THE GRAPH REPRESENTATION OF CONSTRAINTS IN MECHANISMS

Andreas Müller
Uni. of Michigan - Shanghai Jiao Tong Uni. Joint Institute
Shanghai, P.R. China

Offer Shai
Mechanical Engineering School,
Tel-Aviv University
Tel-Aviv, Israel

ABSTRACT

There are two established approaches to represent constraints: the body-bar (BB) and the bar-joint (BJ) graph that can be used in machine theory. They are referred to as topological graphs as they describe the relation between members of a mechanism. It is known, however, that in many cases these graphs are not unique. Hence any method for kinematic analysis or mobility determination that is based on these topological graphs is prone to failures.

In this paper a generalized and unified concept for the representation of constraints in mechanisms is introduced. It is first shown in which situations BB and BJ representations fail to correctly represent the mechanism. The novel constraint graph is then derived starting from the most general model of constrained rigid bodies. It is shown how BB and BJ graphs result as special cases. Therefore the new graph representation is called the 'mixed graph'. It is further shown how this novel mixed constraint graph allows for computation of the correct generic (topological) mobility, and thus overcomes the problems of BB and BJ representations.
Keyword: Mobility, constraint graphs, topological redundancy

1. INTRODUCTION

Methods for determination of topological mobility rely on a graph representation of the mechanism's constraints. As such the topological criteria like the Kutzbach-Grübler formula [1], or the combinatorial pebble game algorithm [2,5,7,8,9,10,17] are well-known. They fail, however, if the constraints are not in one-one correspondence with the mechanism kinematics. Moreover it is known that the established graph representations (body-bar and bar-joint graphs) do not allow for a unique representation of the constraints. This is a sever obstacle for any method that aims to compute the mobility of general mechanisms. Graph representations of mechanisms have been used in various contexts [18,19], and recently it has been used for topology representation of metamorphic linkages [20,21].

The non-uniqueness problem is addressed in this paper. It is shown that the established graph representations can lead to fallacious conclusions about the mobility of a mechanism. To overcome this problem the constraint graphs are derived from the general frame representation. The origin of the BB and BJ is revealed and related to this setting. A novel constraint graph is derived from the general formulation combining BB and BJ. It is called the 'mixed graph' (MG).

A constraint graph is an abstract representation of relations between certain elements that kinematically represent a mechanism. In the known BB and BJ graph these elements represent bodies and frames, respectively.

2. MODELLING CONSTRAINTS IN MECHANISMS

In the most general setting members of a mechanism are represented individually as free rigid bodies in space (or possibly in some subgroup of Euclidean motion depending on the mechanism, such as planar, spherical, or spatial).

Then a rigid body is kinematically represented by reference frame attached to it. Interconnections constraining the motions are modeled by a frame. Hence the frames are constituent modeling elements that allow for unique representation of a general mechanism.

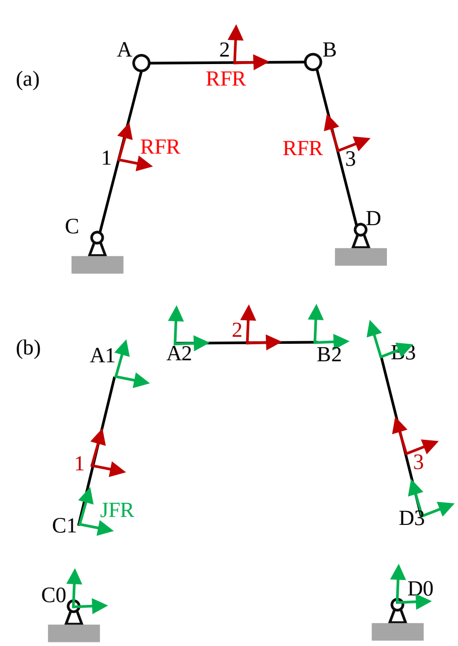

Figure 1: a) 4-bar mechanism, b) introduction of frames as principle kinematic objects

Figure 1a) shows this for a 4-bar mechanism connected to the ground. Each one of the three movable links is represented by a body-fixed reference frame (RFR) as shown in Figure 1b). Upon this general kinematics modeling various constraint graphs can be introduced, as shown in this paper, including BB and BJ graphs. The important, although well-known, fact is that the different constraint systems can have the same mechanical realization.

3 ESTABLISHED CONSTRAINT GRAPHS

3.1 Bar-Joint (BJ) Constraint Graph

A *bar-joint graph* (BJ) is a graph $G(E, V_J)$ whose vertices correspond to points on the mechanism's bodies, and the edges to translation constraints between these points.
That is, BJ graphs can represent mechanisms with revolute joints in 2D, spherical in 3D, and general prismatic joints.
In the mechanism shown in Figure 2a) all the joints are lower pairs. The binary links 1 and 2 impose a distance constraint between the points O1 and A, and A and B, respectively. The slider (B) is treated as a point, and prismatic joint 3 imposes one translation constraint.

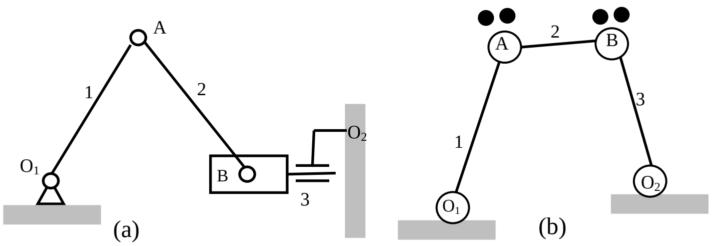

Figure 2: Example of a bar-joint constraint graph of a mechanism. a) The mechanism. b) The corresponding bar-joints graph.

An advantage of the BJ graph is that it allows representing mechanisms having multiple joints (see section 4).

Limitations: The kinematic objects (vertices) used in BJ graphs are points, and thus BJ graphs can only represent distance constraints between points at the bodies of a mechanism. In fact the term 'body-bar' stems from the rigidity theory of structures where each edge represents a mass-less bar imposing a scalar distance constraint between two bodies.

3.2 Body-Bar (BB) Constraint Graph

A *body-bar graph* is graph $G(E, V_B)$ whose vertices correspond to the bodies and the edges represent general scalar constraints between the bodies. In particular, an edge can stand for a distance constraint or a rotation constraint. An edge exists between exactly two bodies and there can be several edges depending on the type of the turning pair. For instance, in the BB graph in Figure 3b) there are two edges between bodies 1 and 2 since the kinematic pair is of type revolute joint. Both constraints account for translational constraints. There is a gear pair, higher pair, between bodies 2 and 3 thus only one edge appears between vertices 2 and 3.
The advantage of BB graphs is that they can represent general constraints between bodies of a mechanism, including higher kinematic pairs.

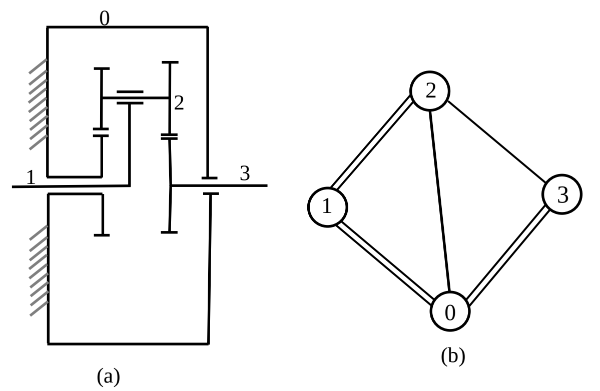

Figure 3: A gear train (a) and its corresponding body-bar graph (b).

Limitations: BB graphs cannot uniquely represent mechanisms with multiple joints (see section 4).

4 THE MULTIPLE JOINT PROBLEM

Many mechanisms comprise several revolute joints whose axes are on a common line or spherical joints with common center of rotation. Such joints are called multiple joints. These joints can be represented uniquely with BJ graphs. However, BJ graphs cannot be used to represent general joints or general constrains. BB graphs on the other hand cannot uniquely represent the constraints of a mechanism if it contains multiple joints since there are different ways to choose the kinematic pairs.

For instance, the mechanism in Figure 4a) has three different BB graphs all corresponding to the same mechanism. In Figure 4b) vertex 2 corresponds to a ternary link, in Figure 4c) and d) vertex 3 and 4, respectively, corresponds to a ternary link. This has consequences for the ability to correctly determine the mobility.

To overcome this problem in the next two sections a novel constraint graph is proposed that combines the advantages of BB and BJ graphs.

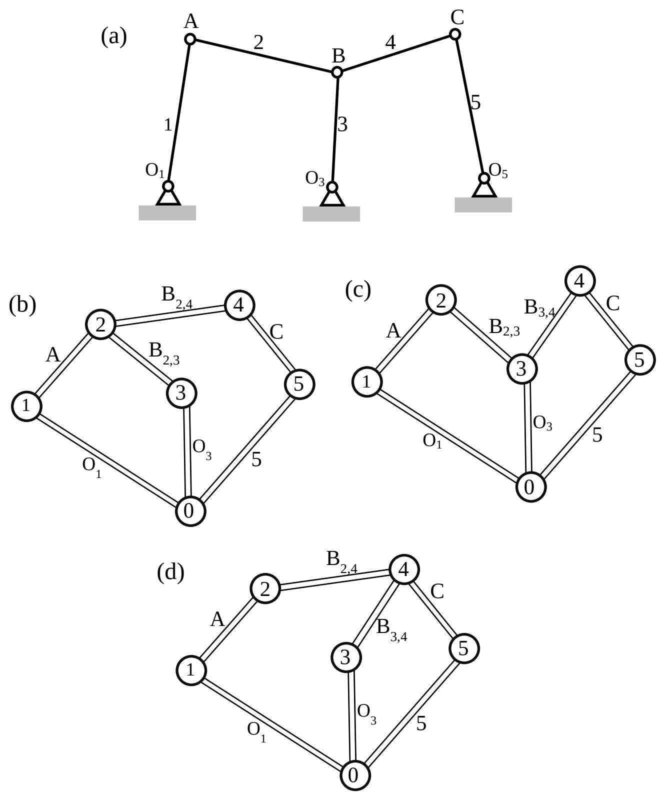

Figure 4: A mechanism (a) and its three different possible BB graphs (b), (c), and (d).

5 GENERAL KINEMATIC CONSTRAINT GRAPHS

5.1 The General Concept

The concept of constraint graphs is generally applicable to represent interrelations within a mathematical model. The basic idea of constraint graphs is to represent the existence of relations between model variables by means of a graph. Moreover the graph representation of relations between objects in mathematical models is the central idea behind non-causal bond-graph modeling approach [22].

In the context of mechanism kinematics the model variables are the DOFs of frames used to model bodies and interactors (joints, contacts, etc.), and the relations are the constraints imposed by the interactors. This is a powerful concept that accounts for holonomic and non-holonomic as well as bilateral and unilateral constraints. The graph only indicates the presence of constraints but not the particular type (translational, rotational).

A general *kinematic constraint graph* is a vertex-labeled undirected graph denoted $G(E, V, peb)$, where peb is a weight function that assigns to a vertex $v \in V$ the weight $peb(v)$. A vertex v represents a kinematic modeling element (generally a frame but can also be a point). The weight is the DOF that is currently assigned to the object represented by vertex v. Initially, when constructing the graph, this is the DOF of the object when it is *unconstrained* (i.e. unconstrained frame or point). That is, the constraint graph in its initial setup represents all frames in the kinematic model together with the number of constraints between them. The number of edges of G is denoted with $e(G) = |E|$, and the number of vertices with $v(G) = |V|$.

Now the constraint graph construction involves two steps:

1. Assigning kinematic objects (frames and points)
2. Introducing the constraints between the kinematic objects

In order to model the kinematics an arbitrary number of (possibly redundant) reference frames and points can be introduced as long as these collectively describe the mechanism's kinematics. In the most general setting a rigid body is kinematically represented by a body-fixed reference frame subjected to certain constraints. A joint connecting two bodies (or a general constraint between the bodies) is represented by a body-fixed frame on each body, and a set of constraints according to the joint mobility. In special cases, like spherical joints in spatial mechanisms and revolute joints in planar mechanisms for instance, the joints do not impose orientation constraints and it is sufficient to introduce joint reference points instead of frames. For example in planar mechanisms the configuration of a frame, i.e. a body, is given by two position coordinates and a rotation angle. In the plane a body can also be represented by two points fixed to it. That is, there are two position coordinates for each point which are subject to one distance constraint, hence only three of the four position coordinates are independent. The same argument applies to three points for bodies in spatial mechanisms. Consequently frames and points can be considered as the building blocks for kinematics modeling, and these are the objects associated to the vertices of the constraint graph. It is important to emphasize the generality of this approach, and that the BB and BJ graphs are just special cases. In fact, if only body-fixed reference frames are used, this leads to the BB, and if only points are used, leads to the BJ representation discussed above.

In order derive the BJ and BB the general graph must be reduced without losing information. To this end a reduction rule is introduced next.

5.2 Reduction Rule for Constraint Graphs

A general constraint graph can be simplified noticing that the DOFs of a vertex that is only connected to two other vertices can be uniquely eliminated.

Reduction rule: Let x be a vertex that has exactly two adjacent vertices, y and z. Let $\text{peb}(x)$ be the number of free pebbles assigned to vertex x and $n_e(u, v)$ stands for the number of edges/constraints between u and v. Then do

1. Remove vertex x and all its incident edges,
2. Add edges between the two neighbors y and z according to the following relation:

$$n_e(y, z) = n_e(y, x) + n_e(x, z) - peb(x).$$

This reduction rule hence determines the constraints between vertices y and z when the mobility of vertex x is incorporated. The significance of this reduction rule is that it allows deriving the well-known BB and BJ graphs as special cases from the general graph, and more importantly it gives rise to a novel mixed graph that will be introduced in the next section. Before three special cases are considered.

Case 1: Bodies and Joints represented by Frames

This is the most general approach. Each body is equipped with a reference frame, and for each joint a frame on the connected bodies is introduced. Figure 1b) shows this for the planar 4-bar mechanism. Each frame constitutes a vertex of the constraint graph. In the plane each unconstrained frame has 3 DOF, i.e. $peb(v) = 3$ for all vertices. The revolute joints impose two constraints. The rigid connection of frames at the same body gives rise to three constraints. The constraint graph is shown in Figure 5. Notice that the weights of vertices are represented by 'pebbles' rather than integer numbers in anticipation of the combinatorial algorithm that will be used for mobility computation [17].
This graph as it stands is a proper representation of the 4-bar kinematics, but it involves an apparent number of redundant frames. Application of the reduction rule to the vertices A1,A2,B2,B3,C1,D1 representing joint frames leads to the BB graph in Figure 6c).
As already discussed the specific characteristics of the BB graph is that the relative configuration of adjacent bodies is constrained by means of joint constraints, and that this relative configuration cannot be uniquely defined for multiple joints.

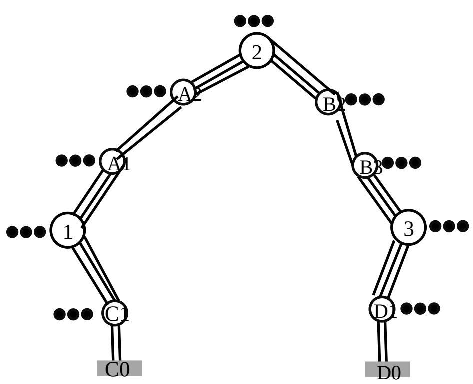

Figure 5: BB graph for the modeling of 4.bar in Figure 1b)

Case 2: Bodies represented by Frames, Joints by Points

The planar 4-bar only comprises revolute joints that for planar mechanisms only impose translation constraints. Hence the joint frames can be replaced by points, and the joints be modeled as constraints between these body-fixed points as shown Figure 6a). The bodies themselves are still represented by reference frames. Figure 6b) shows the corresponding constraint graph. This graph can be reduced applying the reduction step in two different ways:

I) Elimination of vertices representing frames: Applying the reduction step to the frame-vertices yields the BJ constraint graph in Figure 6d), which only comprises point-vertices.

II) Elimination of all joint-vertices: The reduction rule applied to the point vertices A and B yields the BB graph in Figure 6c).

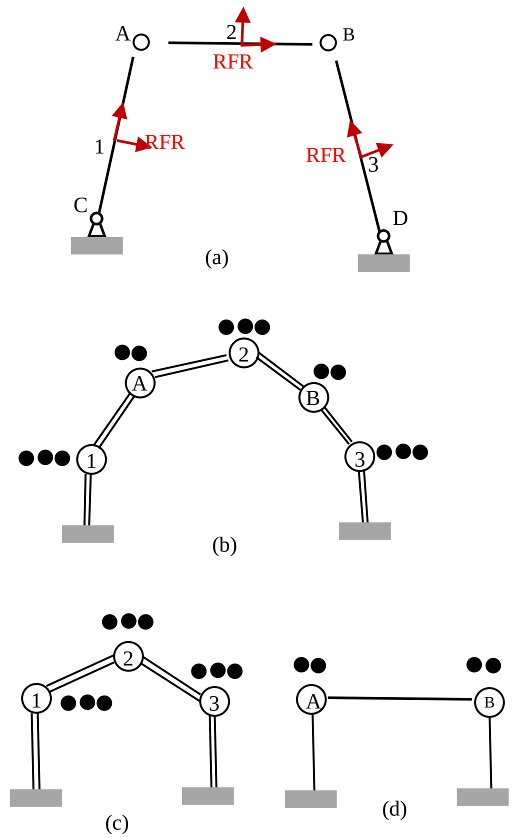

Figure 6: Bodies represented by frames and joints by points. a) The constituent elements of the four-bar link. b) Pebbles associated to each vertex. c) BB graph. d) BJ graph.

Case 3: Joints represented by Points

Since at least two joints are attached to the bodies it is sufficient to represent the joint configurations.
The reduction variant I) in the previous section shows that for revolute joint mechanism the body-fixed reference frames can be omitted and the reference points for a joint be identified leading eventually to the BJ graph. If each body is connected to at least 2 joints in 2D, and 3 joints in 3D, it is sufficient to introduce the joint-vertices and to omit body frames. This is the crucial observation that allows for defining unique constraint graphs for multiple joints.

Apparently when starting from the most general description of unconstrained bodies a number of redundant reference frames/points are introduced which can be eliminated be the reduction rule. On the other hand BB and BJ a priori do not involve such redundant elements. To signify this, the following is introduced:

Definition: A constraint graph is *vertex reducible* if there are two adjacent vertices x, y with either $\text{peb}(x) = n_e(x, y)$ or $\text{peb}(y) = n_e(x, y)$.

As such BB and BJ are not vertex reducible (assuming joints with DOF>0).
Using point-vertices (as in example 3) is only possible if the joints impose position constraints only, i.e. revolute in 2D and spherical in 3D. On the other hand it is only necessary to use reference points for multiple joints, which can only be revolute/spherical.
In conclusion reference points and reference frames can be introduced as deemed appropriate, but reference points must be used to model multiple joints.
This leads to a mixed constraint graph.

6 NOVEL CONSTRAINT GRAPH: MIXED GRAPH (MG)

The foregoing examples show the freedom in representing the mechanism kinematics by various forms of a generalized constraint graph. This generality is a valuable feature, but the most crucial property expected from a constraint representation is that it is unique in the sense that for the introduced kinematic reference objects (frames, points) the constraint system (i.e. the graph) is unique. As discussed in section 4 this is not the case when representing multiple joints by reference frames. In order to ensure uniqueness a mixed constraint graph is introduced as basis for mobility analysis.

Definition: A *mixed constraint graph* is a constraint graph $\mathrm{G} = (\mathrm{V_B} \cup \mathrm{V_J}, \mathrm{E}, \mathrm{peb})$ where multiple joints are represented by reference points. Here $\mathrm{v} \in \mathrm{V_B}$ represents the reference frame of a rigid body, and $\mathrm{v} \in \mathrm{V_J}$ the reference point of a joint.

Notice that in the mixed graph only multiple joints must be modeled by reference points, while other revolute joints can be modeled as desired. The important point why the mixed graph is important is that allows treating multiple joints while it can also uniquely represent the general constraints or joints.

Example: In the example, in Figure 7 the multiple joint B as well as joints A, C, E are modeled by reference points. Clearly with the mixed graph the limitations of the BJ and BB constraint graphs can be overcome by combining these two representations.

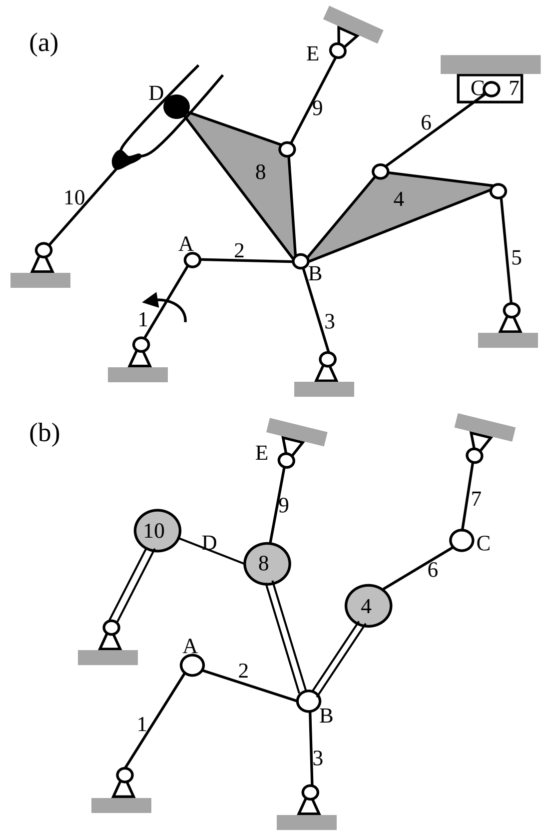

Figure 7: A mechanism (a) and its corresponding mixed constraint graph (b).

7. APPLICATION OF THE MIXED GRAPH

The MG allows for unique representation of the constraints in a mechanism. Hence it provides a graph representation of a mechanism that can be used as basis for combinatorial methods to compute the mobility. Such a method is the pebble game algorithm [1-10]. This algorithm has been developed and there is a variant for BJ and BB graphs. In these forms it cannot be applied directly to the MG representation. Therefor the algorithm must be amended so be able to process a mixture of BJ and BB. A preliminary extension toward such a mixed pebble game was reported in [17,23]. This is not in the scope of this paper and will be reported in a forthcoming publication. In this paper a general approach to constraint modeling has been proposed that will be the basis for such an extended algorithm.

8. CONCLUSIONS

A novel mixed graph representation of constraints in mechanisms has been proposed that overcomes the non-uniqueness problem of the established constraint graph representation. It combines the advantages of the body-bar and bar-joint constraint graphs since it allows representing general constraints (lower pairs, higher pairs, non-holonomic constraints). Hence this graph will allow for a correct determination of mobility using the combinatorial pebble game algorithm. The later can, however, not be applied in its current form but has to be modified to work with the mixed constraint graph. This will be reported in a forthcoming paper. In this paper it is further shown how the established BB and BJ graphs can be constructed as special cases from a general modeling of the mechanism kinematics in terms of body-fixed reference frames.

ACKNOWLEGEMENTS

The second authors would like to thank Israel Science Foundation for their financial support, grant No. 1668/11.

REFERENCES

[1] G. Gogu, Mobility of mechanism: a critical review, Mech. Mach. Theory, 40 (2005) 1068–1097.

[2] D. J. Jacobs and B. Hendrickson. An algorithm for two-dimensional rigidity percolation: the pebble game. Journal of Computational Physics, 137 (1997) 346 – 365

[3] A. Jain, Graph theoretic foundations of multibody dynamics, Part I: structural properties, Multibody. Syst. Dyn. 26 (2011) 307–333.

[4] G. Laman. On graphs and rigidity of plane skeletal structures. J. Engineering Mathematics, 4 (1970) 331–340.

[5] A. Lee and I. Streinu, Pebble game algorithms and sparse graphs, Discrete Mathematics, 308: 8 (2008) 1425-1437.

[6] A. Müller: Generic Mobility of Rigid Body Mechanisms, Mech. Machine Theory, Vol. 44, no. 6, June 2009 1240-1255.

[7] O. Shai, "Rigidity of 2d and 3d Pinned Frameworks and the Pebble Game", London. Mathematical Society Workshop: Rigidity of Frameworks and Applications, July 12-15, Lancaster University , England, 2010.

[8] O. Shai, A. Sljoka and W. Whiteley, "Directed Graphs, Decompositions, and Spatial Rigidity" submitted to Discrete Applied Mathematics, 2010.

[9] A. Sljoka , 2006, Counting for Rigidity, Flexibility and extensions via the Pebble Game Algorithm, Master's Thesis, York University.

[10] A. Sljoka, O Shai and W. Whiteley, "Checking mobility and decomposition of linkages via Pebble Game Algorithm" ASME Design Engineering Technical Conferences, August 28-31, 2011, Washington, USA.

[11] T. Tiong-Seng, Henneberg's method for bar and body frameworks, Structural Topology 17 (1991), 53–58.

[12] K.J. Waldron, The constraint analysis of mechanisms, J. Mech. 1 (1966) 101–114.

[13] K.J. Waldron, A study of overconstrained linkage geometry by solution of closure equations – Part I. Method of study, Mech. Mach. Theory 8 (1973) 95–104.

[14] K.J. Waldron, A study of overconstrained linkage geometry by solution of closure equations – part II. Four-bar linkages with lower pairs other than screw joints, Mech. Mach. Theory 8 (1973) 233–24.

[15] C. Wampler, B.T. Larson, A.G. Erdman, A new mobility formula for spatial mechanisms, in: Proc. ASME International Design Engineering Technical Conferences (IDETC), September 4–7, 2007, Las Vegas.

[16] C.W. Wampler, J.D. Hauenstein, A.J. Sommese: Mechanism mobility and a local dimension test, Mech. Mach. Theory 46 (2011) 1193–1206.

[17] O. Shai O. and A. Muller, Computational Algorithm for Determining the Generic Mobility of Floating Planar and Spherical Linkages, 6th Int. Workshop on Computational Kinematics, May 12-15, Barcelona, Spain, 2013

[18] D. Vucina, F. Freudenstein: An application of graph theory and nonlinear programming to the kinematic synthesis of mechanisms, Mechanism and Machine Theory 26 (6) (1991) 553–563.

[19] Hong-Sen Yan, Yu-Ting Chiu: An algorithm for the construction of generalized kinematic chains, Mechanism and Machine Theory, Vol. 62 (2013), pp. 75–98

[20] D. Li, Z. Zhang, J.M. McCarthy: A constraint graph representation of metamorphic linkages, Mechanism and Machine Theory, Vol. 46, 2011, pp. 228–238

[21] J.S. Dai, J. Rees-Jones: Matrix representation of topological changes in metamorphic mechanisms, ASME Journal of Mechanical Design 127 (4) (2005), pp. 610–619

[22] Karnopp, Dean C., Margolis, Donald L., Rosenberg, Ronald C.: System dynamics: a unified approach, Wiley, 1990

[23] O. Shai, A. Müller: A Novel Combinatorial Algorithm for Determining the Generic/Topological Mobility of Planar and Spherical Mechanisms, 37th Mechanisms and Robotics Conference (MECH), ASME 2013 International Design Engineering Technical Conferences, August 4-7, 2013, Portland, OR

